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FEATURES OF THE CONSTRUCTION OF CORRECTION FUNCTIONS IN THE ANALYSIS OF THE FREE VIBRATIONS OF MECHANICAL SYSTEMS BY BUBNOV'S METHOD†

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Free vibrations of mechanical systems are investigated by Bubnov's method, using coordinate functions that do not satisfy all the boundary conditions of the problem. To eliminate discrepancies in the boundary conditions, correction functions are introduced, orthogonal to all coordinate functions involved in the analysis of the natural vibrations of mechanical systems. This method of constructing correction functions reveals the existence of differences between nearly equal numbers in the frequency equation and in the expressions for the vibrational modes. Moreover, for a given number N of coordinate functions, the method guarantees increased precision in computing all the natural frequencies of the mechanical system up to and including the $(N + 1)$th partial frequency as the number of terms of the power series in the frequency parameter representing the correction functions is increased. In the limit, as the number of terms of the power series tends to infinity, one is assured of obtaining exact values of the frequencies and vibrational modes in the indicated frequency range.

The sources and development of Bubnov's method were investigated in detail in a book by E. I. Grikolyuk [1]. The coordinate functions in Bubnov's method must satisfy all the boundary conditions of the boundary-value problem, and the selection of such a complete system of functions often becomes complicated. An effective way of overcoming this difficulty, by introducing correction functions, was considered in [2, 3].

Below, in the context of analysing the natural vibrations of mechanical systems, it is proposed, unlike in [2, 3], to orthogonalize the correction functions, each of which is represented by a power series in the frequency parameter, with respect to all coordinate functions involved in the computation. It is shown that the number of coordinate functions involved sets an upper limit to the frequency range of the mechanical system that can be investigated. The correction functions eliminate discrepancies in the boundary conditions and increase the precision with which the natural frequencies and vibrational modes can be calculated in the frequency range considered. In the limit, as the number of terms of the power series representing the correction functions tends to infinity, one is assured of obtaining exact values of the frequencies and modes in the system only in that frequency range. When synthesizing the dynamical characteristics of component structures, orthogonalization of the correction functions with respect to the coordinate functions involved in the computations has made it possible to eliminate the "breakdown" of the solution in its computer implementation [4–6].

1. CORRECTION FUNCTIONS IN PROBLEMS OF DYNAMICS

Consider the homogeneous boundary-value problem of determining the natural frequencies and modes of vibration of a mechanical system, written in the form

$$\begin{aligned} &L(u)-\lambda u = 0 \\ &M_i(u(a)) = 0, \quad i = 1, 2, \dots, n; \quad M_i(u(b)) = 0, \quad i = n+1, n+2, \dots, 2n \end{aligned} \tag{1.1}$$

where L is the operator of the boundary-value problem (of order $2n$), λ is the frequency parameter, $u(x)$ is the unknown functions, M_i are the operators of the boundary conditions, and $[a, b]$ is the interval in which the boundary-value problem is defined.

Suppose that, in the interval $[a, b]$, one has a system of coordinate functions $\{\varphi_k\}$ which do not satisfy (or do not satisfy all) the boundary conditions of problem (1.1).

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Following the approach described in [3], we will seek the general solution (not a particular solution, as in the classical Bubnov method) of boundary-value problem (1.1) in the form

$$u = \sum_f(x) + \sum_\varphi(x); \quad \sum_f(x) = \sum_{i=1}^{2n} C_i f_i(x), \quad \sum_\varphi(x) = \sum_{k=1}^{\infty} A_k\varphi_k(x) \tag{1.2}$$

where f_i ($i = 1, 2, \dots, 2n$) are correction functions, introduced in order to eliminate discrepancies in the boundary conditions.

The constants C_i in formula (1.2) are determined from the boundary conditions of the problem, while the coefficients A_k are determined by Bubnov's procedure: solution (1.2) is substituted into the initial equation (1.1), which is then orthogonalized in turn in the interval $[a, b]$ with respect to each of the functions φ_k ($k = 1, 2, \dots, N$) involved in the computation.

Thus, the constants C_i are determined from the boundary conditions of the problem:

$$M_j\left(\sum_f(x)\right) + M_j\left(\sum_\varphi(x)\right) = 0$$
$$(x = a \quad \text{for} \quad j = 1, 2, \dots, n; \quad x = b \quad \text{for} \quad j = n+1, n+2, \dots, 2n) \tag{1.3}$$

The equations for calculating the coefficients A_k have the form

$$\int_a^b\left[L\left(\sum_f(x) + \sum_{j=1}^{\infty} A_j\varphi_j(x)\right) - \lambda\left(\sum_f(x) + \sum_{j=1}^{\infty} A_j\varphi_j(x)\right)\right]\varphi_k(x)dx = 0 \tag{1.4}$$

In practical calculations the number of equations (1.4) is finite: $k = 1, 2, \dots, N$, and the solution obtained is therefore approximate. Equating the determinant of the homogeneous system of algebraic equations to zero, we obtain an equation for the natural frequencies of the mechanical system under consideration. The vibrational modes will be uniquely defined by the values of the coefficients C_i and A_k if an additional normalization condition is introduced.

If the coordinate functions satisfy all the boundary conditions, it follows from Eqs (1.3) that all the C_i vanish. In that case, Eqs (1.4) become the classical system of equations of Bubnov's method, which yields approximate values for the N required lowest frequencies and modes. The differences between the computed values of the frequencies and modes relative to the exact solution increase as the harmonic number of the vibrations increases.

If the coordinate functions are solutions of a boundary-value problem that differs from (1.1) only in its boundary conditions, then each function φ_k has its own partial frequency σ_k. If such coordinate functions are unknown, they and the corresponding partial frequencies may be determined approximately by Bubnov's method from system (1.4) with $C_i = 0$ ($i = 1, 2, \dots, 2n$). In other words, this case may always be realized by changing to new coordinate functions, that is, by a linear transformation in Eqs (1.2)–(1.4).

When that is done, it turns out that the coefficients A_k in Eqs (1.4) are not interconnected. If none of the unknown frequencies coincides with the partial frequencies, Eqs (1.4) can be used to eliminate A_k from Eqs (1.3) and (1.2). If the coordinate functions fail to satisfy only a few boundary conditions of the initial problem, say, one of them, then the number of correction functions equals the number of unsatisfied boundary conditions. Sometimes this method enables one to construct an exact series solution of quite complicated mechanical systems [3].

Following the approach described in [3], the correction functions $f_i(x)$ are represented as power series in the frequency parameter

$$f_i(x) = f_{0i}(x) + \lambda f_{1i}(x) + \dots + \lambda^p f_{pi}(x) \tag{1.5}$$

The functions $f_{0i}(x), a_{1i}(x), \dots, f_{pi}(x)$ are determined from a recurrence sequence of static boundary-value problems, of the form

$$L(f_{0i}) = \sum_{l=1}^{N} B_l\varphi_l(x), \quad M_j(f_{0i}) = 0, \quad j = 1, 2, \dots, 2n; \quad j \neq i$$

$$M_i(f_{0i}) = 1; \quad \int_a^b f_{0i}\varphi_l(x)dx = 0, \quad l = 1, 2, \ldots, N$$
$$L(f_{ki}) = f_{(k-1)i}, \quad M_j(f_{ki}) = 0, \quad j = 1, 2, \ldots, 2n; \quad k = 1, 2, \ldots, p \tag{1.6}$$

It will be recalled that in the boundary conditions the argument of the functions takes a specific value ($x = a$ or $x = b$).

The correction functions in Eqs (1.6) are orthogonal to all coordinate functions involved in the computation, but in [3] the correction functions were orthogonalized only to coordinate functions with zero partial frequencies, which describe the displacement of the mechanical system as a rigid body.

The first functions f_{0i} in formulae (1.5) eliminate discrepancies in the boundary conditions and, together with the subsequent functions f_{ki} ($k = 1, 2, \ldots, p$), accelerate the convergence of the series with respect to the coordinate functions in the solution obtained.

Suppose the coordinate functions $\{\varphi_k\}$ are solutions of a boundary-value problem of the form (1.1) (the basic problem) with one boundary condition changed. To fix our ideas, let us assume that the unsatisfied boundary condition has the form

$$M_i[u(b)] = 0 \tag{1.7}$$

In that case [3], it will suffice to introduce a single correction function $f(x)$. Then the solution of problem (1.1) may be written in the form

$$u = Cf(x) + \sum_{k=1}^{\infty} A_k\varphi_k(x) \tag{1.8}$$

Only one equation remains in system (1.3)

$$CM_i(f(b)) + \sum_{k=1}^{\infty} A_k M_i[\varphi_k(b)] = 0 \tag{1.9}$$

Equations (1.4) become

$$a_k(\sigma_k - \lambda)A_k - Ca_kB_k = 0, \quad k = 1, 2, \ldots, N$$
$$a_k(\sigma_k - \lambda)A_k - C\left(\frac{\lambda}{\sigma_k}\right)^{p+1} M_i[\varphi_k(b)] = 0, \quad k = N+1, N+2, \ldots \tag{1.10}$$

If the unknown frequencies are not identical with any of the partial frequencies σ_k, Eqs (1.10) may be used to eliminate A_k from Eqs (1.8) and (1.9)

$$u = C\left[f_i(x) + \sum_{k=1}^{N} \frac{B_k\varphi_k(x)}{\sigma_k - \lambda} + \sum_{k=N+1}^{\infty} \left(\frac{\lambda}{\sigma_k}\right)^{p+1} \frac{M_i[\varphi_k(b)]\varphi_k(x)}{a_k(\sigma_k - \lambda)}\right]$$
$$C\left[M_i(f_i(b)) + \sum_{k=1}^{N} \frac{B_kM_i[\varphi_k(b)]}{\sigma_k - \lambda} + \sum_{k=N+1}^{\infty} \left(\frac{\lambda}{\sigma_k}\right)^{p+1} \frac{M_i^2[\varphi_k(b)]}{a_k(\sigma_k - \lambda)}\right] = 0 \tag{1.11}$$

The equation for computing the natural frequencies of the system is obtained by equating the bracketed expression in the second equation of (1.11) to zero.

Each of the N first coordinate functions is represented in Eqs (1.11) by terms proportional to $(\sigma_k - \lambda)^{-1}$, which is independent of the number of terms of the power series in the correction function (1.5).

If $\sigma_k = 0$, the corresponding term cannot be represented by any power series, and therefore it is necessary to orthogonalize the correction function to the coordinate functions describing the motion of the basic system (in the terminology of [3]) as a rigid body.

But if $\sigma_k \neq 0$, we have:

for $\lambda < \sigma_k$

$$\frac{1}{\sigma_k-\lambda} = \sum_{l=0}^{\infty}\frac{\lambda^l}{\sigma_k^{l+1}} = \sum_{l=0}^{p}\frac{\lambda^l}{\sigma_k^{l+1}} + \left(\frac{\lambda}{\sigma_k}\right)^{p+1}\sum_{l=p+1}^{\infty}\frac{\lambda^l}{\sigma_k^{l+1}} = \sum_{l=0}^{p}\frac{\lambda^l}{\sigma_k^{l+1}} + \left(\frac{\lambda}{\sigma_k}\right)^{p+1}\frac{1}{\sigma_k-\lambda} \tag{1.12}$$

for $\lambda > \sigma_k$

$$\frac{1}{\sigma_k-\lambda} = -\sum_{l=0}^{\infty}\frac{\sigma_k^l}{\lambda^{l+1}} = -\sum_{l=0}^{p}\frac{\sigma_k^l}{\lambda^{l+1}} + \left(\frac{\sigma_k}{\lambda}\right)^{p+1}\sum_{l=p+1}^{\infty}\frac{\sigma_k^l}{\lambda^{l+1}} = -\sum_{l=0}^{p}\frac{\sigma_k^l}{\lambda^{l+1}} + \left(\frac{\sigma_k}{\lambda}\right)^{p+1}\frac{1}{\sigma_k-\lambda} \tag{1.13}$$

Note that for any finite values of p in formulae (1.12) and (1.13), if $\lambda \neq \sigma_k$, we have the following relations

$$\frac{1}{\sigma_k-\lambda} = \sum_{l=0}^{p}\frac{\lambda^l}{\sigma_k^{l+1}} + \left(\frac{\lambda}{\sigma_k}\right)^{p+1}\frac{1}{\sigma_k-\lambda} = -\sum_{l=0}^{p}\frac{\sigma_k^l}{\lambda^{l+1}} + \left(\frac{\sigma_k}{\lambda}\right)^{p+1}\frac{1}{\sigma_k-\lambda} \tag{1.14}$$

As p is increased, provided the restrictions imposed on the value of λ are satisfied, the second terms in (1.12) and (1.13) decrease in absolute value, and, beginning at some value of p, they become negligibly small compared with the first terms; in the limit as $p \to \infty$ the second terms tend to zero.

But if λ exceeds the limits stipulated for each equation, the infinite power series in formulae (1.12) and (1.13) will diverge, and taking the sum of the first p terms of the divergent series (as in (1.14)) leads to the appearance of differences of two nearly equal numbers. Neglecting the second terms in (1.14) at "irregular" values of λ leads to a crude error: both terms in the sum are similar in absolute value but of opposite sign. The larger p (and thereby the farther λ is from the boundaries of the "regular" domain), the closer the absolute values of the two terms. Naturally, under these conditions, it is impossible to take the limit as $p \to \infty$. In real computations, however, "breakdown" of the solution may occur at a finite value of p.

Thus, the process proposed here, to orthogonalize the correction functions with respect to the N first coordinate functions, has the result that in the frequency range $0 \le \lambda < \sigma_{N+1}$ the power series (1.5) converges as $p \to \infty$ to a value characterizing the "contribution" to the solution of the remaining coordinate functions φ_k ($k = N+1, N+2, \ldots$), so that the last (third) terms in brackets in formulae (1.11) disappear. In computer calculations, the possibility of neglecting these terms when computing the lower frequencies and vibrational modes of mechanical systems is implemented at a finite fairly small value of the degree p of the polynomial (this value of p decreases as the number N of coordinate functions involved in the calculation increases).

2. THE LONGITUDINAL VIBRATIONS OF A ROD

Let us consider the problem of computing the natural frequencies and modes of longitudinal vibrations of a homogeneous cantilever rod, which are determined by solving the following boundary-value problem

$$\frac{d^2u}{d\alpha^2} + \lambda^2 u = 0; \quad u(0) = \frac{du(1)}{d\alpha} = 0 \quad \left(\alpha = \frac{x}{l}, \lambda^2 = \frac{m\omega^2 l^2}{EF}\right) \tag{2.1}$$

where l is the length of the rod, m is its mass per unit length, EF is the tensile and compressive stiffness, ω is the angular velocity of the vibrations, and x is the longitudinal coordinate of sections of the rod measured from the lower clamped end.

Problem (2.1) has an exact solution which is also obtained by Bubnov's method if one takes the coordinate functions to be $\{\sin(k-1/2)\pi\alpha\}$ ($k = 1, 2, \ldots, \infty$)

$$\lambda_k = (k-1/2)\pi; \quad u_k = \sin(k-1/2)\pi\alpha, \quad k = 1, 2, \ldots \tag{2.2}$$

If one takes the complete system of coordinate functions $\{\varphi_k\} = \{\sin k\pi\alpha\}$ ($k = 1, 2, \ldots$) (the vibrational modes of a rod with attached ends) which fail to satisfy only one boundary condition, a single correction function is needed. The solution of the problem may be represented in the form

$$u = C(f_0(\alpha) + \lambda^2 f_1(\alpha) + \ldots + \lambda^{2t} f_t(\alpha)) + \sum_{k=1}^{n_1} A_k \sin k\pi\alpha + \sum_{k=n_1+1}^{\infty} A_k \sin k\pi\alpha \tag{2.3}$$

The functions $f_0(\alpha), f_1(\alpha), \dots, f_i(\alpha)$ are solutions of the sequence of static boundary-value problems

$$\frac{d^2 f_0}{d\alpha^2} = -\sum_{k=1}^{n_1} B_k \sin k\pi\alpha; \quad f_0(0) = 0; \quad f_0'(1) = 1; \quad \int_0^1 f_0 \sin k\pi\alpha d\alpha = 0, \quad k = 1, 2, \dots, n_1$$
$$\frac{d^2 f_j}{d\alpha^2} = -f_{j-1}; \quad f_j(0) = 0, \quad f_j(1) = 0, \quad j = 1, 2, \dots, t \tag{2.4}$$

The first two terms of the correction function, determined from boundary-value problems (2.4), take the form

$$f_0 = \frac{2}{2n_1+1}\left(\frac{\alpha}{2} + \sum_{k=1}^{n_1} \frac{(-1)^k \sin k\pi\alpha}{k\pi}\right), \quad f_1 = \frac{2}{2n_1+1}\left(\frac{\alpha-\alpha^3}{12} + \sum_{k=1}^{n_1} \frac{(-1)^k \sin k\pi\alpha}{(k\pi)^3}\right) \tag{2.5}$$

If we confine our attention to the single term f_0 in the correction function, the solution of the initial problem may be written in the form

$$u(\alpha) = C_1\left\{\frac{1}{2n_1+1}\left(\alpha + 2\sum_{k=1}^{n_1} \frac{(-1)^k \sin k\pi\alpha}{k\pi}\right) + \right.$$
$$\left. + \frac{2}{2n_1+1}\sum_{k=1}^{n_1} \frac{(-1)^k k\pi \sin k\pi\alpha}{\lambda^2 - (k\pi)^2} + \frac{2\lambda^2}{2n_1+1}\sum_{k=n_1+1}^{\infty} \frac{(-1)^k \sin k\pi\alpha}{k\pi(\lambda^2 - (k\pi)^2)}\right\} \tag{2.6}$$

The frequencies of the vibrations are determined from the equation

$$1 + \frac{2}{2n_1+1}\sum_{k=1}^{n_1} \frac{(k\pi)^2}{\lambda^2 - (k\pi)^2} + \frac{2\lambda^2}{2n_1+1}\sum_{k=n_1+1}^{\infty} \frac{1}{\lambda^2 - (k\pi)^2} = 0 \tag{2.7}$$

For a two-term approximation of the correction function $f_0 + \lambda^2 f_1$, the solution of problem (2.1) may be written in the form

$$u(\alpha) = \frac{C_1}{2n_1+1}\left\{\left(\alpha + 2\sum_{k=1}^{n_1} \frac{(-1)^k \sin k\pi\alpha}{k\pi}\right) + \lambda^2\left(\frac{\alpha}{6} - \frac{\alpha^3}{6} + 2\sum_{k=1}^{n_1} \frac{(-1)^k \sin k\pi\alpha}{(k\pi)^3}\right) + \right.$$
$$\left. + 2\sum_{k=1}^{n_1} \frac{(-1)^k k\pi \sin k\pi\alpha}{\lambda^2 - (k\pi)^2} + 2\lambda^4 \sum_{k=n_1+1}^{\infty} \frac{(-1)^k \sin k\pi\alpha}{(k\pi)^3(\lambda^2 - (k\pi)^2)}\right\} \tag{2.8}$$

The corresponding frequency equation is obtained by substituting expression (2.8) into the boundary condition at the free end

$$1 + \frac{2}{2n_1+1}\left\{\lambda^2\left(-\frac{1}{3} + \sum_{k=1}^{n_1} \frac{1}{(k\pi)^2}\right) + \sum_{k=1}^{n_1} \frac{(k\pi)^2}{\lambda^2 - (k\pi)^2} + \lambda^4 \sum_{k=n_1+1}^{\infty} \frac{1}{(k\pi)^2(\lambda^2 - (k\pi)^2)}\right\} = 0 \tag{2.9}$$

The arbitrary factor C_1 in Eqs (2.6) and (2.8) is completely determined by introducing some normalization condition for the vibrational modes.

In the limiting case when $n_1 = 0$ the solutions (2.6), (2.7) and (2.8), (2.9) become the solutions obtained in [2].

In the interval $(0, (n_1 + 1)\pi)$ the correction series in powers of the frequency parameter converges when the number of terms increases without limit. Therefore, given a specific value of n_1, the last term in Eq. (2.9) may be ignored when the number of terms in the correction series is increased, when one is looking for frequencies in the interval $(0, (n_1 + 1)\pi)$.

The first natural frequency of the cantilever rod ($\lambda_1 = \pi/2$) lies in the convergence domain of the correction power series, and therefore, when seeking it, one can disregard all the coordinate functions.

Writing the first four terms of the correction series as determined from boundary-value problems (2.4) with $n_1 = 0$, we have

$$f_0 = \alpha,\ f_1 = \frac{\alpha}{6} - \frac{\alpha^3}{6},\ f_2 = \frac{\alpha^5}{120} - \frac{\alpha^3}{36} + \frac{7\alpha}{360},\ f_3 = -\frac{\alpha^7}{5040} + \frac{\alpha^5}{720} - \frac{7\alpha^3}{2160} + \frac{31\alpha}{15120} \tag{2.10}$$

The solution of boundary-value problem (2.1) may be written in the form

$$u(\alpha) = C_1\left\{\alpha + \lambda^2\left(\frac{\alpha}{6} - \frac{\alpha^3}{6}\right) + \lambda^4\left(\frac{\alpha^5}{120} - \frac{\alpha^3}{36} + \frac{7\alpha}{360}\right) + \right.$$

$$\left. + \lambda^6\left(-\frac{\alpha^7}{5040} + \frac{\alpha^5}{720} - \frac{7\alpha^3}{2160} + \frac{31\alpha}{15120}\right)\right\} \text{(mode of vibration)} \tag{2.11}$$

$$1 - \frac{\lambda^2}{3} - \frac{\lambda^4}{45} - \frac{2\lambda^2}{945} = 0 \text{ (frequency equation)} \tag{2.12}$$

The unique positive real root of Eq. (2.12) is $\lambda_1 = 1.732051$, taking two terms of the correction series into consideration, $\lambda_1 = 1.600720$, taking three terms, and $\lambda_1 = 1.577660$, taking four terms; the exact root is $\lambda_1 = \pi/2$.

Computations of the three lowest modes of vibration of a cantilever rod using Eqs (2.7) and (2.9) were carried out on a computer with double precision. Taking one coordinate function into consideration, the first natural frequency is determined from Eq. (2.9), correct to three decimal places, but from Eq. (2.7) with an error of more than 15%. When ten coordinate functions are taken into consideration, Eq. (2.9) yields four correct decimal places for λ_1 and three for the frequencies λ_2 and λ_3. The error in the three lowest frequencies computed from Eq. (2.7), taking ten coordinate functions into consideration is ~2%. When one hundred coordinate functions are taken into consideration one has six or seven correct decimal places in the values of the required frequencies from Eq. (2.9) and only two from Eq. (2.7).

Expansion of the exact expression for the mode of vibration with respect to the coordinate functions chosen above leads to a rather slowly converging series

$$u_i(\alpha) = \sum_{k=1}^{\infty} \frac{(-1)^k k\pi \sin k\pi\alpha}{\lambda_i^2 - (k\pi)^2} \tag{2.13}$$

Evaluating the derivative of this expression at the point $\alpha = 1$, one obtains a diverging series. A finite member of terms of that series appear in both equations (2.7) and (2.9). The orthogonalization procedure detects the special features of the solution: the presence of small differences (the second term in (2.9)) and of poorly converging series (the third term in (2.9)). These features actually appear in a masked form in the solutions of [2, 3].

3. THE VIBRATIONS OF A ROD ON ELASTIC SUPPORTS

The natural frequencies and modes of longitudinal vibrations of a rod with elastic supports at intermediate sections are determined by solving the following boundary-value problem

$$\frac{d^2u}{d\alpha^2} + \lambda^2 u = 0;\quad \frac{du(0)}{d\alpha} = \frac{du(1)}{d\alpha} = 0,\quad \left.\frac{du}{d\alpha}\right|_{\alpha_k-0}^{\alpha_k+0} = C_k u(\alpha_k),\quad u\Big|_{\alpha_k-0}^{\alpha_k+0} = 0$$
$$\left(\alpha = \frac{x}{l},\quad \lambda^2 = \frac{m\omega^2 l^2}{EF},\quad \bar{C}_k = \frac{C_k l}{EF}\right) \tag{3.1}$$

(C_k is the stiffness of the support at the section $\alpha = \alpha_k$; the remaining notation is the same as in Section 2).

As a complete system of coordinate functions, we choose the longitudinal vibrational modes of a free (unsupported) rod, that is $\{\varphi_i\} = \{\cos i\pi\alpha\}$ $(i = 0, 1, 2, \dots, \infty)$. Corresponding to these modes are the following natural frequencies and reduced masses

$$\sigma_i^2 = (i\pi)^2, \quad a_i = \int_1^0 \varphi_i^2 da = \frac{1}{2}; \quad i = 0, 1, 2, \ldots$$

Since the coordinate functions do not satisfy a dynamic condition at the supports, the solution of the initial problem may be represented in the form

$$u(\alpha) = \sum_{k=1}^{N} D_k f_k(\alpha) + \sum_{i=0}^{p} A_i \cos i\pi\alpha + \sum_{i=p+1}^{\infty} A_i^* \cos i\pi\alpha \tag{3.2}$$

The correction functions, the number of which is equal to the number of intermediate supports, are solutions of the following static boundary-value problems

$$\frac{d^2 f_k}{d\alpha^2} = -\sum_{i=0}^{p} D_{ki} \cos i\pi\alpha$$

$$\frac{df_k(-0)}{d\alpha} = 0, \quad \frac{df_k(1+0)}{d\alpha} = 0, \quad \left.\frac{df_k(\alpha)}{d\alpha}\right|_{\alpha_k-0}^{\alpha_k+0} = 1, \quad f_k(\alpha)\big|_{\alpha_k-0}^{\alpha_k+0} = 0 \tag{3.3}$$

$$\int_0^1 f_k \cos i\pi\alpha d\alpha = 0; \quad k = 1, 2, \ldots, N; \quad i = 0, 1, 2, \ldots, p$$

The sum of the form $\sum_{i=0}^{p} D_{kt}$ in Eqs (3.3) cannot vanish, since the first coordinate function describes the longitudinal displacement of a rigid rod and the corresponding natural frequency is zero [2, 3].

The solution of boundary-value problem (3.3) may be written in the form

$$f_k(\alpha) = f_k^*(\alpha) + 2\sum_{i=1}^{p} \frac{\cos i\pi\alpha_k \cos i\pi\alpha}{(i\pi)^2}$$

$$f_k^*(\alpha) = \begin{cases} -\frac{1}{2}(\alpha^2 + \alpha_k^2) + \alpha_k - \frac{1}{3}, & 0 \le \alpha \le \alpha_k - 0 \\ -\frac{1}{2}[\alpha^2 + 1]^2 + \alpha_k^2 + \frac{1}{6}, & \alpha_k + 0 \le \alpha \le 1 \end{cases} \tag{3.4}$$

Substituting expression (3.2) into Eq. (3.1) and applying Bubnov's orthogonalization procedure to the resulting expression, relative to the coordinate functions, one obtains equations for the unknowns A_i. The coefficients D_k are evaluated from the dynamic conditions at the supports

$$A_0 = \frac{1}{\lambda^2}\sum_{k=1}^{N} D_k, \quad A_i(\lambda^2 - (i\pi)^2) = 2\sum_{k=1}^{N} D_k \cos i\pi\alpha_k, \quad i = 1, 2, \ldots, p$$

$$A_i^*(\lambda^2 - (i\pi)^2) = 2\lambda^2 \sum_{k=1}^{N} D_k \frac{\cos i\pi\alpha_k}{(i\pi)^2}, \quad i = p+1, p+2, \ldots \tag{3.5}$$

$$D_j = \bar{C}_j\left\{\sum_{k=1}^{N} D_k f_k(\alpha_k) + \sum_{i=1}^{p} A_i \cos i\pi\alpha_k + \sum_{i=p+1}^{\infty} A_i^* \cos i\pi\alpha_k\right\}$$

The condition that the determinant of system (3.5) should vanish yields an equation for computing the vibrational frequencies of the rod with intermediate supports.

When there is one support at the section $\alpha = \alpha_k$, Eqs (3.5) become

$$A_0 = \frac{D_1}{\lambda^2}, \quad A_i(\lambda^2 - (i\pi)^2) = 2D_1 \cos i\pi\alpha_k, \quad i = 1, 2, \ldots, p$$

$$A_i^*(\lambda^2 - (i\pi)^2) = \frac{2\lambda^2}{(i\pi)^2} D_1 \cos i\pi\alpha_k, \quad i = p+1, p+2, \ldots \tag{3.6}$$

$$D_1 = \bar{C}_k\left\{D_1\left[-\alpha_k^2+\alpha_k-\frac{1}{3}+2\sum_{i=1}^{p}\frac{\cos^2 i\pi\alpha_k}{(i\pi)^2}\right]+\sum_{i=0}^{p}A_i\cos i\pi\alpha_k+\sum_{i=p+1}^{\infty}A_i\cos i\pi\alpha_k\right\}$$

If none of the frequencies of the supported rod is a frequency of the unsupported rod, the unknowns A_0, A_i, A_i^* are eliminated from the last equation of (3.6) and the following equation is obtained

$$D_1\left\{\bar{C}_1\left[-\alpha_k^2+\alpha_k-\frac{1}{3}+2\sum_{i=1}^{p}\frac{\cos^2 i\pi\alpha_k}{(i\pi)^2}+\frac{1}{\lambda^2}+2\sum_{i=1}^{p}\frac{\cos^2 i\pi\alpha_k}{\lambda^2-(i\pi)^2}+\right.\right.$$
$$\left.\left.+2\lambda^2\sum_{i=p+1}^{\infty}\frac{\cos^2 i\pi\alpha_k}{(i\pi)^2[\lambda^2-(i\pi)^2]}\right]-1\right\} = 0 \tag{3.7}$$

If $p = 0$, this equation implies the frequency equation obtained in [2].

The frequency equation for a supported rod follows from the condition that the expression in braces in Eq. (3.7) should vanish. If the support is absolutely stiff and is placed at $\alpha_k = 0$, the frequency equation is written in the form

$$-\frac{1}{3}+2\sum_{i=1}^{p}\frac{1}{(i\pi)^2}+\frac{1}{\lambda^2}+2\sum_{i=1}^{p}\frac{1}{\lambda^2-(i\pi)^2}+2\lambda^2\sum_{i=p+1}^{\infty}\frac{1}{(i\pi)^2[\lambda^2-(i\pi)^2]} = 0 \tag{3.8}$$

Using the known sums of the series for any p in the limit as $i \to \infty$, one obtains Eq. (3.8) in a form identical with the exact solution

$$\frac{\mathrm{ctg}\lambda}{\lambda} = 0 \Rightarrow \cos\lambda = 0 \tag{3.9}$$

4. THE BENDING VIBRATIONS OF A BEAM

The frequencies and modes of bending vibrations of a beam supported at its ends by hinges and at an intermediate cross-section on an elastic support are determined by solving the following boundary-value problem

$$\frac{d^4w}{d\alpha^4}-\lambda^2 w = 0$$
$$\left.\frac{d^3w}{d\alpha^3}\right|_{\alpha_k-0}^{\alpha_k+0} = \bar{C}_{yk}w(\alpha_k),\quad w\Big|_{\alpha_k-0}^{\alpha_k+0} = \left.\frac{dw}{d\alpha}\right|_{\alpha_k-0}^{\alpha_k+0} = \left.\frac{d^2w}{d\alpha^2}\right|_{\alpha_k-0}^{\alpha_k+0} = 0$$
$$w(0) = w(1) = \frac{d^2w(0)}{d\alpha^2} = \frac{d^2w(1)}{d\alpha^2} = 0 \tag{4.1}$$
$$\left(\alpha = \frac{x}{l},\quad \lambda^2 = \frac{m\omega^2 l^4}{EJ},\quad \bar{C}_{yk} = \frac{C_{yk}l^3}{EJ}\right)$$

where m, EJ and l are the mass per unit length, the bending stiffness and the length of the beam, w is the deflection of the beam, and C_{yk} is the stiffness per unit length of the support at the section $\alpha = x_k/l$.

Suppose the coordinate functions are the vibrational modes of a beam supported by hinges at its ends, which do not satisfy the dynamic condition at the elastic support and are characterized by the relations

$$a_{qi} = 1/2,\quad \lambda_{qi}^2 = (i\pi)^4,\quad \eta_i(\alpha) = \sin i\pi\alpha \tag{4.2}$$

The solution of boundary-value problem (4.1) is represented in the form

$$w = C_1 f + \sum_{i=1}^{n_1} A_i \sin i\pi\alpha + \sum_{i=n_1+1}^{\infty} A_i^* \sin i\pi\alpha \tag{4.3}$$

A correction function f orthogonal to the first n_1 coordinate functions is a solution of the following boundary-value problem

$$\begin{gathered}
\frac{d^4 f}{d\alpha^4} = \sum_{i=1}^{n_1} B_i \sin i\pi\alpha \\
\left.\frac{d^3 f}{d\alpha^3}\right|_{\alpha_k-0}^{\alpha_k+0} = -1, \quad f\Big|_{\alpha_k-0}^{\alpha_k+0} = \left.\frac{df}{d\alpha}\right|_{\alpha_k-0}^{\alpha_k+0} = \left.\frac{d^2 f}{d\alpha^2}\right|_{\alpha_k-0}^{\alpha_k+0} = 0 \\
f(0) = f(1) = \frac{d^2 f(0)}{d\alpha^2} = \frac{d^2 f(1)}{d\alpha^2} = 0 \\
\int_0^1 f \sin i\pi\alpha d\alpha = 0 \quad \text{for} \quad i = 1, 2, \ldots, n_1
\end{gathered} \tag{4.4}$$

The solution of the boundary-value problem (4.4) may be written in the form

$$\begin{gathered}
f = f^* + 2\sum_{i=1}^{n_1}\left(\frac{\sin i\pi\alpha_k}{(i\pi)^4} - \frac{\alpha_k(\alpha_k-1)}{3i\pi}\cos i\pi\alpha_k\right)\sin i\pi\alpha \\
f^* = \begin{cases} -\dfrac{\alpha_k(\alpha_k-1)(\alpha_k-2)}{6}\alpha - \dfrac{(\alpha_k-1)}{6}\alpha^3, & 0 \le \alpha \le \alpha_k - 0 \\ -\dfrac{\alpha_k(\alpha_k-1)(\alpha_k+1)}{6}(\alpha-1) - \dfrac{\alpha_k}{6}(\alpha-1)^3, & \alpha_k + 0 \le \alpha \le 1 \end{cases}
\end{gathered} \tag{4.5}$$

A system of algebraic equations in A_i is obtained by substituting expressions (4.3) and (4.5) into the first equation of (4.1) and orthogonalizing the resulting expression with respect to the coordinate functions $\{\eta_i\} = \{\sin i\pi\alpha\}$ $(i = 1, 2, \ldots, \infty)$

$$\begin{gathered}
A_i[(i\pi)^4 - \lambda^2] + 2C_1\left[\sin i\pi\alpha_k - \frac{\alpha_k(\alpha_k-1)(i\pi)^3}{3}\cos i\pi\alpha_k\right] = 0, \quad i = 1, 2, \ldots, n_1 \\
A_i^*[(i\pi)^4 - \lambda^2] + 2\lambda^2 C_1\left[\frac{\sin i\pi\alpha_k}{(i\pi)^4} - \frac{\alpha_k(\alpha_k-1)}{3(i\pi)}\cos i\pi\alpha_k\right] = 0, \quad i = n_1+1, n_1+2, \ldots
\end{gathered} \tag{4.6}$$

The coefficient C_1 is determined from the boundary condition at the support

$$\begin{gathered}
C_1 = \bar{C}_{yk}\left[C_1\left(-\frac{\alpha_k^2(\alpha_k^2-1)}{3} + 2\sum_{i=1}^{n_1}\left(\frac{\sin i\pi\alpha_k}{(i\pi)^4} - \frac{\alpha_k(\alpha_k-1)}{3i\pi}\cos i\pi\alpha_k\right)\right) + \right. \\
\left. + \sum_{i=1}^{n_1} A_i \sin i\pi\alpha_k + \sum_{i=n_1+1}^{\infty} A_i^* \sin i\pi\alpha_k\right]
\end{gathered} \tag{4.7}$$

Equations (4.6) and (4.7) form a system of homogeneous algebraic equations for the natural frequencies and vibrational modes.

From the condition for the determinant of this system to be zero we obtain the frequency equation for a beam with intermediate support.

The solution in the form (4.6), (4.7) enables us to compute all the frequencies and vibrational modes for any position of the elastic support (including the frequencies and modes with the node at the section where the intermediate support is placed). To determine the frequencies and vibrational modes of the beam with intermediate support that are not identical with any of the frequencies and modes of an unsupported beam, the frequency equation may be transformed to a more transparent form, eliminating the coordinates A_i and A_i^* with the help of Eqs (4.6) from expressions (4.3) and (4.7).

Let us put

$$\chi_i(\alpha_k) = \frac{\sin i\pi\alpha_k}{(i\pi)^4} - \frac{\alpha_k(\alpha_k - 1)}{3i\pi}\cos i\pi\alpha_k$$

The frequency equation can then be written in the form

$$1 - \bar{C}_{yk}\left[\underline{\left(-\frac{\alpha_k^2(\alpha_k-1)^2}{3} + 2\sum_{i=1}^{n_1}\chi_i(\alpha_k)\sin i\pi\alpha_k\right)} - \underline{\underline{2\sum_{i=1}^{n_1}\frac{(i\pi)^4}{(i\pi)^4-\lambda^2}\chi_i(\alpha_k)\sin i\pi\alpha_k}} - 2\lambda^2\sum_{i=n_1+1}^{\infty}\frac{1}{(i\pi)^4-\lambda^2}\chi_i(\alpha_k)\sin i\pi\alpha_k\right] = 0 \quad (4.8)$$

The vibrational modes are computed from the formulae

$$w = C_1\left[f - 2\sum_{i=1}^{n_1}\frac{(i\pi)^4}{(i\pi)^4-\lambda^2}\chi_i(\alpha_k)\sin i\pi\alpha - 2\lambda^2\sum_{i=n_1+1}^{\infty}\frac{1}{(i\pi)^4-\lambda^2}\chi_i(\alpha_k)\sin i\pi\alpha\right] \quad (4.9)$$

The small difference (the singly underlined term in (4.8)) and the segment of a poorly convergent series (the doubly underlined term in (4.8)) are the realities of the solution, which have become obvious when using the procedure for orthogonalizing the correction functions with respect to the coordinate functions.

In conclusion, one should note that differences of nearly equal numbers and functions may be replaced by infinite series. For example, in Eq. (2.9)

$$-\frac{1}{3} + \sum_{k=1}^{n_1}\frac{1}{(k\pi)^2} = -\sum_{k=n_1+1}^{\infty}\frac{1}{(k\pi)^2} \quad (4.10)$$

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